Precession of Pericenter: A More Accurate Approach

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Abstract

In this paper we study the orbits of massive bodies moving in the spacetime generated by a spherically symmetric and non-rotating distribution of mass. More specifically, our treatment discusses the more accurate calculation of the precession of pericenter due to general-relativistic effects. Our result is accurate up to terms of second order, while the precession met in the bibliography is accurate only up to first-order terms.

1 The Schwarzschild spacetime

The first solution of Einstein's field equations was published by Karl Schwarzschild in 1916. It is a solution that informs us about the spacetime generated in the exterior of a spherically symmetric and non-rotating distribution of mass. As is well known, that distribution of mass provides the region that surrounds it with a static, spherically symmetric spacetime. That kind of spacetime is mathematically denoted by the line element (cf. [1])

$$\begin{aligned} ds^2 &= g_{tt}(r)dt^2 + g_{rr}(r)dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2 \\ &= -e^{2\Phi}dt^2 + e^{2\Lambda}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2 \end{aligned}$$

in spherical polar coordinates (t, r, θ, ϕ). In the second step above we introduced the functions $\Phi \equiv \Phi(r)$ and $\Lambda \equiv \Lambda(r)$ in place of the two unknowns $g_{tt}(r)$ and $g_{rr}(r)$ respectively. That replacement was possible since $g_{tt} < 0$ and $g_{rr} > 0$ anywhere in spacetime.[1] Of course, we have to impose a couple of boundary conditions to the aforementioned line element, i.e.

$$\lim_{r\to\infty} \Phi(r) = \lim_{r\to\infty} \Lambda(r) = 0$$

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[1]Bear in mind, however, that this allegation breaks down in the case of black holes, where we should reconsider our system of coordinates.

for we demand, as is physically reasonable, that spacetime be flat far away from the distribution of mass.

The previous results give us the ability to calculate the components $G^{\mu\nu}$ ($\mu, \nu = 0, 1, 2, 3$) of Einstein's tensor. The missing step, now, in order to calculate the unknown functions $\Phi(r)$ and $\Lambda(r)$, is to find out the components $T^{\mu\nu}$ of the stress-energy tensor, plug them into Einstein's field equations,

$$G^{\mu\nu} = 8\pi T^{\mu\nu}$$

(note that here and hereafter we use geometrized units unless otherwise mentioned) and solve the resulting differential equations for $\Phi(r)$ and $\Lambda(r)$. The aforementioned tedious calculations lead us to the expressions[2]

$$e^{2\Phi} = e^{-2\Lambda} = 1 - \frac{2M}{r}$$

where M is the total mass of the distribution of mass. Therefore, we are finally in position to write down the line element in its final form:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2$$

2 Precession of Pericenter

In order to calculate the precession of the pericenter of the elliptic orbit of a massive body revolving an attractive center, we begin from the equation of motion

$$\left(\frac{dr}{d\tau}\right)^2 = \tilde{E}^2 - \left(1 - \frac{2M}{r}\right)\left(1 + \frac{\tilde{L}^2}{r^2}\right) \tag{1}$$

where we suppose that the elliptic orbit is described by the radial distance, r, and the azimuthal angle, ϕ, τ is the proper time, $\tilde{E}$ and $\tilde{L}$ are the energy and momentum per unit mass respectively and M is the total mass of the attractive center. The shape of the effective potential $\tilde{V}^2(r) = \left(1 - \frac{2M}{r}\right)\left(1 + \frac{\tilde{L}^2}{r^2}\right)$ for a typical value of the parameter $\tilde{L}^2$ is shown in Fig. 1. Our study will be limited to energies $\tilde{E}^2 \approx 1$, for we want the body to be in a stable orbit around the attractive center. As an example we can see the points A and B in Fig. 1, among which the body can move in a nearly elliptic orbit.

In order to write equation (1) in the more convenient form $dr/d\phi = f(r)$ we use the fact that

$$\frac{d\phi}{d\tau} \equiv v^\phi = \frac{p_\phi}{m} = g^{\phi\phi}\frac{p_\phi}{m} = \frac{1}{r^2}\tilde{L}$$

[2] For further details see [1]

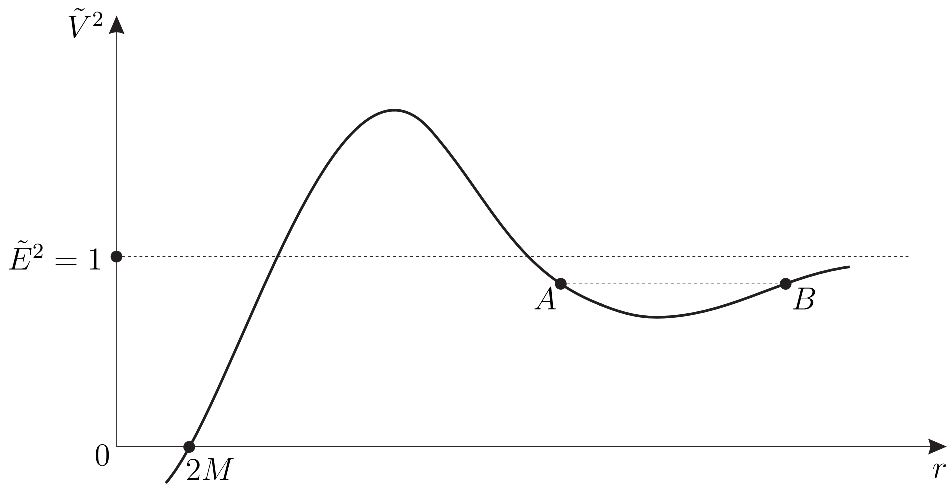

Fig. 1: Typical effective potential of a body with defined angular momentum in a Schwarzschild spacetime

Therefore, we get

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{\tilde{E}^2 - \left(1 - \frac{2M}{r}\right)\left(1 + \frac{\tilde{L}^2}{r^2}\right)}{\frac{\tilde{L}^2}{r^4}}$$

We now proceed with the transformation $u = \frac{r_0}{r}$, where r_0 is a constant with length dimensions. That transformation gives us a differential equation with no dimensions:

$$\frac{du}{d\phi} = \sqrt{\frac{\tilde{E}^2 - 1}{\tilde{L}^2} r_0^2 + \frac{2Mr_0}{\tilde{L}^2} u - u^2 + \frac{2M}{r_0} u^3}$$

It is clear that we can separate the variables in the last differential equation, thus taking

$$\frac{du}{\sqrt{\frac{\tilde{E}^2-1}{\tilde{L}^2} r_0^2 + \frac{2Mr_0}{\tilde{L}^2} u - u^2 + \frac{2M}{r_0} u^3}} = d\phi$$

Let, now, a, b and c be the roots of the polynomial

$$P(u) = \frac{\tilde{E}^2 - 1}{\tilde{L}^2} r_0^2 + \frac{2Mr_0}{\tilde{L}^2} u - u^2 + \frac{2M}{r_0} u^3$$

Then, $P(u)$ can be written as

$$P(u) = \alpha(u - a)(b - u)(c - u)$$

where $\alpha = 2M/r_0$. Because of the equality of the coefficients of the same-order terms for the two forms of $P(u)$ we get the equations

$$abc = \frac{1-\tilde{E}^2}{2M\tilde{L}^2} r_0^3 \tag{2}$$

$$ab + c(a+b) = \frac{r_0^2}{\tilde{L}^2} \tag{3}$$

and

$$a + b + c = \frac{r_0}{2M} \tag{4}$$

We suppose that within the range of values of energy for which we study our problem, $P(u)$ has the three real roots, a, b and c for which $a < b \ll c$. Therefore, for $a \le u \le b$ we can state that the relation $c \gg u$ holds. That statement enables us to Taylor-expand the term $1/\sqrt{c-u}$,

$$\frac{1}{\sqrt{c-u}} \approx \frac{1}{\sqrt{c}}\left(1 + \frac{u}{2c}\right)$$

and get

$$\frac{du}{\sqrt{\alpha(u-a)(b-u)(c-u)}} = d\phi \Rightarrow$$

$$\Rightarrow \frac{1}{\sqrt{\alpha c}} \frac{du}{\sqrt{(u-a)(b-u)}} + \frac{1}{2c\sqrt{\alpha c}} \frac{u\, du}{\sqrt{(u-a)(b-u)}} = d\phi$$

We integrate the last relation for u from a to b, so ϕ varies from zero to $\phi_{\text{final}}/2$. The calculations yield

$$\frac{1}{\sqrt{\alpha c}} \int_a^b \frac{du}{\sqrt{(u-a)(b-u)}} + \frac{1}{2c\sqrt{\alpha c}} \int_a^b \frac{u\, du}{\sqrt{(u-a)(b-u)}} = \int_0^{\phi_{\text{final}}/2} d\phi \Rightarrow$$

$$\Rightarrow \frac{1}{\sqrt{\alpha c}}\pi + \frac{1}{2c\sqrt{\alpha c}} \frac{a+b}{2}\pi = \frac{\phi_{\text{final}}}{2} \tag{5}$$

for $\int_a^b \frac{du}{\sqrt{(u-a)(b-u)}} = \arctan\left(\frac{u - \frac{a+b}{2}}{\sqrt{(u-a)(b-u)}}\right)\Bigg|_{u=a}^{u=b} = \pi$ and $\int_a^b \frac{u\, du}{\sqrt{(u-a)(b-u)}} = \frac{a+b}{2}\pi$.

Let, now, $a + b = \epsilon \ll c$. From equation (4) we get $c = \frac{r_0}{2M} - \epsilon$ and so

$$\frac{1}{\sqrt{\alpha c}} \approx 1 + \frac{M\epsilon}{r_0} + \frac{3}{2}\frac{M^2\epsilon^2}{r_0^2}$$

Therefore, equation (5) becomes

$$\phi_{\text{final}} \approx 2\pi + 2\pi\frac{M\epsilon}{r_0} + 3\pi\frac{M^2\epsilon^2}{r_0^2} + \frac{\epsilon}{2c}\pi\left(1 + \frac{M\epsilon}{r_0} + \frac{3}{2}\frac{M^2\epsilon^2}{r_0^2}\right)$$

where we neglect terms of order higher than $(M\epsilon/r_0)^2$. Apparently, the resulting precession is

$$\Delta\phi = \phi_{\text{final}} - 2\pi$$

$$\Delta\phi \approx 2\pi\frac{M\epsilon}{r_0} + 3\pi\frac{M^2\epsilon^2}{r_0^2} + \frac{\epsilon}{2c}\pi\left(1 + \frac{M\epsilon}{r_0} + \frac{3}{2}\frac{M^2\epsilon^2}{r_0^2}\right)$$

$$\Delta\phi \approx 2\pi\frac{M\epsilon}{r_0} + 3\pi\frac{M^2\epsilon^2}{r_0^2} + \frac{\epsilon}{2\left(\frac{r_0}{2M} - \epsilon\right)}\pi\left(1 + \frac{M\epsilon}{r_0} + \frac{3}{2}\frac{M^2\epsilon^2}{r_0^2}\right)$$

$$\Delta\phi \approx 2\pi\frac{M\epsilon}{r_0} + 3\pi\frac{M^2\epsilon^2}{r_0^2} + \pi\frac{M\epsilon}{r_0}\left(1 + \frac{2M\epsilon}{r_0} + \frac{4M^2\epsilon^2}{r_0^2}\right)\left(1 + \frac{M\epsilon}{r_0} + \frac{3}{2}\frac{M^2\epsilon^2}{r_0^2}\right)$$

$$\Delta\phi \approx 3\pi\frac{M\epsilon}{r_0} + 6\pi\frac{M^2\epsilon^2}{r_0^2}$$

$$\Delta\phi \approx 3\pi\frac{M\epsilon}{r_0}\left(1 + \frac{2M\epsilon}{r_0}\right) \tag{6}$$

where we Taylor-expanded the term $\left(\frac{r_0}{2M} - \epsilon\right)^{-1} = \frac{2M}{r_0}\left(1 - \frac{2M\epsilon}{r_0}\right)^{-1}$ keeping, one more time, terms of order not higher than $(M\epsilon/r_0)^2$.

In order to calculate ϵ we will use equations (2) and (4), from which we get

$$\left(\frac{r_0}{2M} - \epsilon\right)\epsilon + \frac{1 - \tilde{E}^2}{2M\tilde{L}^2}r_0^3\frac{1}{\frac{r_0}{2M} - \epsilon} = \frac{r_0^2}{\tilde{L}^2}$$

Again, if we use the Taylor expansion of the term $\left(\frac{r_0}{2M} - \epsilon\right)^{-1}$ which appears in the last equation, we take

$$\left[1 - \frac{4M^2(1 - \tilde{E}^2)}{\tilde{L}^2}\right]\epsilon^2 - \left(\frac{r_0}{2M} + \frac{1 - \tilde{E}^2}{\tilde{L}^2}2Mr_0\right)\epsilon + \frac{r_0^2}{\tilde{L}^2}\tilde{E}^2 = 0$$

The straightforward solution of the last equation gives us the values

$$\epsilon = \frac{\frac{r_0}{2M} + \frac{1-\tilde{E}^2}{\tilde{L}^2}2Mr_0}{2\left[1 - \frac{4M^2(1-\tilde{E}^2)}{\tilde{L}^2}\right]} \pm \frac{\frac{r_0}{2M}\sqrt{1 - \frac{16M^2}{\tilde{L}^2} + \frac{24M^2}{\tilde{L}^2}(1 - \tilde{E}^2) + \frac{48M^4}{\tilde{L}^4}(\tilde{E}^2 + \frac{1}{3})(1 - \tilde{E}^2)}}{2\left[1 - \frac{4M^2(1-\tilde{E}^2)}{\tilde{L}^2}\right]}$$

From the two possible values of ϵ (positive and negative sign) we choose the one with the negative sign. That choice is based on the fact that we need ϵ to be a small compared to $\frac{r_0}{2M}$ (cf. equation (4) and the discussion that follows it). The negative sign validates the requirement $\epsilon \ll c \approx \frac{r_0}{2M}$, something that is not accomplished via the use of the positive sign. Therefore, that choice and the fact that for $\tilde{E}^2 \approx 1$ it is $\left(\tilde{E}^2 + \frac{1}{3}\right)(1 - \tilde{E}^2) \approx \frac{4}{3}(1 - \tilde{E}^2)$, enable us to write that

$$\epsilon \approx \frac{\frac{r_0}{2M} + \frac{1-\tilde{E}^2}{\tilde{L}^2} 2Mr_0 - \frac{r_0}{2M}\sqrt{1 - \frac{16M^2}{\tilde{L}^2} + \frac{24M^2}{\tilde{L}^2}(1 - \tilde{E}^2) + \frac{64M^4}{\tilde{L}^4}(1 - \tilde{E}^2)}}{2\left[1 - \frac{4M^2(1-\tilde{E}^2)}{\tilde{L}^2}\right]}$$

If we Taylor-expand the square root above and keep terms of order not higher than $\frac{M^4}{\tilde{L}^4}$, we get

$$\epsilon \approx \frac{r_0}{4M}\frac{1 + \frac{1-\tilde{E}^2}{\tilde{L}^2}4M^2 - \left[1 + \frac{4M^2}{\tilde{L}^2}(1 - 3\tilde{E}^2) + \frac{16M^4}{\tilde{L}^4}(3 - 5\tilde{E}^2)\right]}{1 - \frac{4M^2(1-\tilde{E}^2)}{\tilde{L}^2}}$$

where we neglect the term $\frac{72M^4}{\tilde{L}^4}(1 - \tilde{E}^2)^2$ which appears during the calculations since $\tilde{E}^2 \approx 1$. Finally, we Taylor-expand the term $\left[1 - \frac{4M^2(1-\tilde{E}^2)}{\tilde{L}^2}\right]^{-1}$ and we end up with

$$\epsilon \approx \left[\frac{2M\tilde{E}^2}{\tilde{L}^2} - \frac{8M^3}{\tilde{L}^4}(\tilde{E}^2 - 3)\left(\tilde{E}^2 - \frac{1}{2}\right)\right] r_0 \tag{7}$$

If we replace the acquired value of ϵ to equation (6) we get the final result

$$\Delta\phi = \frac{6\pi M^2}{\tilde{L}^2}\left[\left(1 + \frac{14M^2}{\tilde{L}^2}\right)\tilde{E}^2 - \frac{6M^2}{\tilde{L}^2}\right] \tag{8}$$

in terms of the total mass of the attractive center and the energy and momentum per unit mass.

Moreover, we can express the precession found (equation (8)) as a function of the eccentricity, e, and the semi-major axis, a, of the elliptic orbit of the body. If we assume that the mass of the attractive center is much larger than the mass of the orbiting body, then

$$\tilde{L}^2 = Ma(1 - e^2) \qquad \text{and} \qquad \tilde{E}^2 = 1 - M/2a$$

Therefore the precession in terms of e and a is given by the formula

$$\Delta\phi = \frac{6\pi M}{a(1 - e^2)}\left[1 - \frac{M}{2a}\left(1 - \frac{16}{1 - e^2}\right)\right] \tag{9}$$

where we neglect terms of order higher than $(M/a)^2$.

Equation (9) can be used to calculate the precession of the perihelion of planet Mercury's elliptic orbit around the Sun. The semi-major axis of Mercury's orbit is $a = 5.791016 \times 10^{10}$ m, its eccentricity is $e = 0.205615$, while the mass of the Sun is $M = 1.9892 \times 10^{30}$ kg. Of course, in order to find some real numerical values we should write down equation (9) in SI units. Equation (9) as we see it is valid as long as we remember that we use geometrized units, where the speed of light, c, and the constant of gravitational attraction, G, are taken to be equal to one: $c = G = 1$. The form of equation (9) in the SI system of units is

$$\Delta\phi = \frac{6\pi GM}{ac^2(1-e^2)}\left[1 - \frac{GM}{2ac^2}\left(1 - \frac{16}{1-e^2}\right)\right]$$

where $G = 6.672599 \times 10^{-11} \mathrm{N} \cdot \mathrm{m}^2/\mathrm{kg}^2$ and $c = 299792458$ m/s. So if we take into account the fact that Mercury spins around the Sun 415 times within a century, then we get the precession

$$\Delta\phi_{\mathrm{Mercury}} = 42.964926'' \text{ per century}$$

Note that in common bibliography the term $1 - \frac{GM}{2ac^2}\left(1 - \frac{16}{1-e^2}\right)$—whose numerical value is 1.000004813 in the case of Mercury—is just equal to unity. That exlains the fact that the precession we meet in the bibliography is $\Delta\phi_{\mathrm{Mercury}} = 42.964720''$ per century. It is clear that the precession derived by this new approach for the case of planet Mercury does not differ much from the precession derived with the usual way [1]. However, if we happen to study the precession of pericenter for an orbit near a strong gravitational field (e.g. in the vicinity of a black hole), then the extra term is expected to produce additional precession that will definitely be significant.

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References

[1] Bernard F. Schutz, *A First Course in General Relativity*, 1985, Cambridge University Press